Problem-solving: Dimensional analysis 1

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Introduction

A **quantity** is something measurable. It could be a mass, or a speed, or a force, and so on. The quantity will be measured in **units**. This gives us a standard measure to compare things to, so that we can then compare them to each other. The **dimension** of a quantity tells us what *kind* of thing it is, or what kind of things it can be reduced to. We indicate the dimension of a quantity using square brackets, so [mass] means "the dimension of mass".

We often use combinations of units to quantify a quantity. For example, we often use miles and hours to quantify our speed, rather than having a whole new unit of its own. (You could, of course, define one if you wanted to.) As another example, we can give power in terms of Watts, or we can give it in terms of $\mathrm{J\,s^{-1}}$.

In the same way, we can use combinations of dimensions to give the basic dimensions of a quantity. You can pick whatever set of dimensions you like to be your basic dimensions, but the standard choice is:

$$\begin{aligned}[\text{mass}] &= M\\ [\text{length}] &= L\\ [\text{time}] &= T\end{aligned}$$

It's important that your basic dimensions are independent of one another. If we had chosen [length], [time] and [speed] as our three dimensions, we would never actually need [speed], because

$$[\text{speed}] = [\text{length}]/[\text{time}]$$

so we could always rewrite it in those terms. If we want to write speed in terms of its basic dimensions, we would have $[\text{speed}] = LT^{-1}$. Similarly for acceleration a:

$$[a] = \frac{[v]}{[t]} = \frac{LT^{-1}}{T} = LT^{-2}$$

where v is speed and t is time.

If you want a fuller set of basic units, you may also wish to add

$$\begin{aligned}[\text{current}] &= I \\ [\text{temperature}] &= \Theta\end{aligned}$$

Then you're covered for pretty much all physics you might want to do.

You can be very rigorous about dimensional analysis, but you can also just use the ideas it incorporates to sense-check your physics and maths. Over the next couple of workshops, we'll do a bit of both.

Using formulas to do dimensional analysis

Formulas tell us how quantities are related in the real world - for example, how much voltage you need to push a certain amount of current through a certain resistance ($V = IR$). That means formulas also tell us how the dimensions of those quantities are related, because if quantities are equal, their dimensions must be equal too.

$$\begin{aligned}V &= IR \\ [V] &= [IR] \\ [V] &= [I] \times [R]\end{aligned}$$

This means we can use formulae to help us with dimensional analysis.

Worked example: what is voltage?

What are the dimensions of voltage, in terms of our basic dimensions, M, L, T and I?

Solution

Let's start with one of the most memorable formulas that uses voltage:

$$\begin{aligned}[V] &= [I] \times [R] \\ &= I \times [R]\end{aligned}$$

(Remember, I is one of our basic dimensions.)

So, this now boils down to a question of the dimensions of resistance. Obviously $V = IR$ can't get us out of this one - we'd just end up with $V = V$. What else do we know about resistance? How about its relation to power and current, $P = I^2R$? So $R = P/I^2$. What does that give us in terms of dimensions? Well, power is the amount of energy used or given out (E) in a given time (t), and current is already one of our allowed dimensions.

$$\begin{aligned}[R] &= \left[\frac{P}{I^2}\right] \\ &= \frac{[P]}{[I^2]} \\ &= \frac{\left[\frac{E}{t}\right]}{[I]^2} \\ &= \frac{[E]}{[t] \times I^2} \\ &= \frac{[E]}{TI^2} \\ &= [E]T^{-1}I^{-2}\end{aligned}$$

Now we need to find a way to break down energy, as it's not one of our basic dimensions. What formulae do we know that relate energy to things like mass, length and time? How about we think about gravitational potential energy? In that case, $E = mgh$ where m is the mass of our object, g is the gravitational constant and h is the height it is at. Remember that the gravitational constant is actually just an acceleration (think about free-fall SUVAT problems). Considering the dimensions, then:

$$\begin{aligned}[E] &= [mgh] \\ &= [m] \times [g] \times [h] \\ &= M \times LT^{-2} \times L \\ &= ML^2T^{-2}\end{aligned}$$

Let's use this to get the dimensions for resistance:

$$\begin{aligned}[R] &= [E]T^{-1}I^{-2} \\ &= ML^2T^{-2}T^{-1}I^{-2} \\ &= ML^2T^{-3}I^{-2}\end{aligned}$$

And in turn, we can use this to get our dimensions for voltage:

$$\begin{aligned}[V] &= I[R] \\ &= IML^2T^{-3}I^{-2} \\ &= ML^2T^{-3}I^{-1}\end{aligned}$$

So in terms of the units,

$$1\,\mathrm{V} = 1\,\frac{\mathrm{kg\,m^2}}{\mathrm{s^3\,A}}$$

or equivalently, a volt is a mass times a distance times an acceleration divided by a current. Who knew?!

Problems

- What are the basic dimensions of force? What is 1 N in kg, m and s?
- How many other ways can you find that give you the basic dimensions of energy?
- What are the dimensions for the spring constant in Hooke's law?
- What are the basic dimensions of capacitance? [1]

[1] Capacitance is a measure of an object's ability to hold charge. It depends on things like the size, shape and material of the object, so it is a physical property of the object independent of things like what kind of electrical circuit it is in. The "standard" capacitor is two metal plates held a fixed distance apart. With a certain voltage V, you can put a certain amount Q of charge on the capacitor. The capacitance is defined as how much charge that voltage can put into the capacitor, the charge per volt, so $C = Q/V$. For a particular capacitor, increasing the voltage will mean you can put more charge on it - the ratio of the charge and voltage (the capacitance) is the thing that's fixed.